

AD 37652

MAXIMUM DIAMETER OF ABSTRACT POLYTOPES

BY

ILAN ADLER and GEORGE B. DANTZIG

TECHNICAL REPORT NO. 71-12

AUGUST 1971

OPERATIONS
RESEARCH
HOUSE



Stanford
University
CALIFORNIA

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
Springfield, Va. 22151

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION
DEPARTMENT OF OPERATIONS RESEARCH STANFORD UNIVERSITY STANFORD, CALIF		2b. GROUP
3. REPORT TITLE		
Maximum Diameter of Abstract Polytopes		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)		
TECHNICAL REPORT		
5. AUTHOR(S) (Last name, first name, initial)		
Ilan Adler and George B. Dantzig		
6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1971	25	4
8a. CONTRACT OR GRANT NO.	8a. ORIGINATOR'S REPORT NUMBER(S)	
N00014-67-A-0112-0011	71-12	
b. PROJECT NO.	8b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
NR-047-064		
d.		
10. AVAILABILITY/LIMITATION NOTICES		
This document has been approved for public release and sale; its distribution is unlimited		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY	
	Operations Research Program Code 434 Office of Naval Research Washington, D. C. 20360	
13. ABSTRACT		
<p>Walkup and Klee studied the diameter of ordinary convex polytopes which is defined as the smallest integer k such that all pairs of vertices can be joined by a path of k or less neighboring vertices. The well known d-step (or Hirsch) conjecture for d dimensional polytopes with n facets states that the maximum diameter is $n - d$. Walkup and Klee showed the conjecture as correct for all $n - d \leq 5$. In an effort to extend the results, the authors of this paper abstract (in the form of three simple axioms) some of the combinatorial properties of ordinary polytopes and show that these are sufficient to establish the maximum diameter is $n - d$ for all $n - d \leq 5$ and a variety of bounds for other values of n and d. Abstract polytopes are a special class of pseudo manifolds. The results provide insight into the number of iterations required to solve a linear program using the simplex method.</p>		

DD FORM 1473
1 JAN 64

UNCLASSIFIED

Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
	<p>Convex Sets</p> <p>Polytopes</p> <p>Hirsch Conjecture</p> <p>d-Step Conjecture</p> <p>Polytope Diameter</p> <p>Abstract Polytope Diameter</p>					

INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (corporate author) issuing the report.
- 2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.
- 2b. **GROUP:** Automatic downgrading is specified in DoD Directive S200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.
3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parentheses immediately following the title.
4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.
5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.
6. **REPORT DATE:** Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.
- 7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.
- 7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.
- 8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.
- 8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.
- 9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.
- 9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers (either by the originator or by the sponsor), also enter this number(s).
10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those

imposed by security classification, using standard statements such as:

- (1) "Qualified requesters may obtain copies of this report from DDC."
- (2) "Foreign announcement and dissemination of this report by DDC is not authorized."
- (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."
- (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."
- (5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring (paying for) the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical content. The assignment of links, roles, and weights is optional.

MAXIMUM DIAMETER OF ABSTRACT POLYTOPES

by

Ilan Adler

and

George B. Dantzig

Technical Report No. 71-12

August 1971

DEPARTMENT OF OPERATIONS RESEARCH

Stanford University

Stanford, California

Research and reproduction of this report was partially supported by Office of Naval Research, Contract N-00014-67-A-0112-0011; U.S. Atomic Energy Commission, Contract AT[04-3]326 PA #18; National Science Foundation, Grant GP 25738.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

This document has been approved for public release and sale; its distribution is unlimited.

MAXIMUM DIAMETER OF ABSTRACT POLYTOPES

by

Ilan Adler and George B. Dantzig

1. Abstract polytope--definition and notation.

Given a finite set T of symbols, a family P of subsets of T (called vertices) forms a d -dimensional abstract polytope if the following three axioms are satisfied:

- (i) Every vertex of P has cardinality d .
- (ii) Any subset of $d-1$ symbols of T is either contained in no vertices of P or in exactly two (called neighbors or adjacent).
- (iii) Given any pair of vertices $v, \bar{v} \in P$, there exists a sequence of vertices $v = v_0, \dots, v_k = \bar{v}$ such that
 - (a) v_i, v_{i+1} are neighbors ($i = 0, \dots, k-1$)
 - (b) $\{v \cap \bar{v}\} \subset v_i$ ($i = 0, \dots, k$).

It is convenient to delete from T all symbols that are not used to define vertices. Hence we denote $UP = \{Uv | v \in P\}$.

Let u be a subset of P such that $|u| = k$ ($|u|$ denotes the cardinality of u). If $P' = \{v \in P | v \supset u\}$ is nonempty we say that P' is the face of P which is generated by u and denote it by $F_P(u)$ or simply $F(u)$ if the abstract polytope P is clear. It is not difficult to verify that the family $\{v-u | v \in F_P(u)\}$ of subsets obtained by deleting u from each vertex of such a face is a $(d-k)$ -dimensional abstract polytope. In the sequel we shall use this property of faces extensively. Whenever we mention a face as an abstract polytope, it is to be understood that the deleting

of common symbols has been performed. Since $F_P(u)$ corresponds to a $(d-k)$ -dimensional abstract polytope we say that it is a $(d-k)$ -dimensional face of P . Zero, one and $d-1$ dimensional faces are called, respectively, vertices, edges, and facets.

A d -dimensional abstract polytope with n facets is called an (n,d) -abstract polytope. (Note that $n = |U_P|$.) We denote by $\mathcal{P}(n,d)$ the class of all (n,d) -abstract polytopes.

The graph $G(P)$ of an abstract polytope P is the graph whose vertices and edges correspond 1-1 to the vertices and edges of P , respectively.

Note that axiom (iii) is satisfied by P if, and only if, the graph of every face of P is connected, and that if we augment P by including all subsets of the vertices of P , then axioms (i)-(iii)(a) define a $(d-1)$ -dimensional pseudo-manifold (with no boundary).

2. Relation between abstract and simple polytopes.

Abstract polytopes are (combinatorially) closely related to simple polytopes. A simple polytope can be expressed as the set of solutions of a bounded and non-degenerate linear program [4]. Suppose the latter consists of m equations, in n non-negative variables whose coefficient matrix is of rank m . One can associate n symbols with the index set of the n columns of the coefficient matrix. Then the family of subsets of symbols which correspond to the non-basic columns of all the basic feasible solutions (i.e., vertices) of the linear program forms an (n,d) -abstract polytope where $d = n-m$.

This is true because any feasible solution is defined uniquely by the subset of $d = n - m$ non-basic variables set to zero (axiom (i)). Given a basic feasible solution, a new basic solution can be obtained by dropping any one of the d non-basic variables. Exactly one of the basic variables can be set equal to zero in its place (under non-degeneracy and boundness). This generates a neighboring vertex (axiom (ii)). Given any two vertices v and \bar{v} , then by restricting ourselves to the lowest dimensional face common to v and \bar{v} (i.e., holding at zero value the subset of non-basic variables common to the two vertices), a path of neighboring vertices from v to \bar{v} can be found (e.g., by using the simplex method and a suitably chosen objective function) (axiom (iii)).

Although the class of abstract polytopes includes (combinatorially) that of simple polytopes the converse is not true. Indeed by a theorem of Steinitz (see [2]) the graph of every 3-dimensional abstract polytope is planar. However, the graph of the 3-dimensional abstract polytope displayed in Figure 2 is easily shown to be non-planar. Hence no simple polytope can have the graph structure of this particular abstract polytope.

3. Paths and diameters.

Let P be an abstract polytope and let $v, \bar{v} \in P$. A path of length k from v to \bar{v} in P is a sequence of vertices $v = v_0, \dots, v_k = \bar{v}$ such that v_i, v_{i+1} are neighbors ($i = 0, \dots, k-1$). (Note that vertices of the path are not required to be in $F_P(v \cap \bar{v})$.) The diameter $\delta(P)$ of P is the smallest integer k such that any two vertices of P can be joined by

a path of length less than or equal to k . We denote by $\Delta_a(n,d)$ the maximum of $\delta(P)$ over all (n,d) -abstract polytopes. This corresponds to Klee and Walkup's $\Delta_b(n,d)$ for ordinary simple polytopes [1]. In general, of course, $\Delta_a(n,d) \geq \Delta_b(n,d)$.

Our main objective is to establish values and bounds for $\Delta_a(n,d)$. We shall show in particular that the unsolved d -step (or Hirsch) conjecture (that $\Delta_a(n,d) \leq n-d$) holds for $n-d \leq 5$ thus paralleling results of Klee and Walkup [1] for ordinary polytopes. Our arguments, however, are based on fewer axioms and imply theirs as a special case.

4. Some preliminary results.

We shall make frequent use of the following theorem:

Theorem 1. (Adler, Dantzig, Murty [3]) Given an abstract polytope P , if two vertices v, \bar{v} in P do not have a symbol (say A) in common then there exists an "A-avoiding path" joining them; i.e., there exists a path from v to \bar{v} such that no vertex along the path contains A .

The next Theorem is the analog of a result of Klee and Walkup in [1]. The proof here is similar.

Theorem 2. For $k = 0, 1, 2, \dots$

- (i) $\Delta_a(n,d) \leq \Delta_a(n+k, d+k)$
- (ii) $\Delta_a(n,d) \leq \Delta_a(n+k, d)$
- (iii) $\Delta_a(n,d) \leq \Delta_a(n+2k, d+k) - k.$
- (iv) $\Delta_a(2d,d) = \Delta_a(d+k, k), k \geq d.$

Proof. We shall prove (i)-(iii) for $k = 1$, the extension to $k > 1$ is trivial.

Let P be an (n, d) -abstract polytope such that $\delta(P) = \Delta_a(n, d)$.

(i) Let $A \in UP$ and let $A' \notin UP$ be a new symbol, define P' as an abstract polytope identical with P except the symbol A' replaces A . Define \tilde{P} as a new abstract polytope with vertices $v \cup A'$ and $v' \cup A$ for all $v \in P$ and all $v' \in P'$.

It is easy to verify that \tilde{P} is an $(n+1, d+1)$ -abstract polytope with a diameter at least as big as $\delta(P)$, thus

$$\Delta_a(n, d) = \delta(P) \leq \delta(\tilde{P}) \leq \Delta_a(n+1, d+1)$$

(ii) Let $A' \notin UP$ be a new symbol and $v' \in P$. Let v_1, \dots, v_d be the d subsets of v' with cardinality $d-1$. Define $\tilde{P} = (P - v') \cup \bigcup_{i=1}^d (v_i \cup A')$.

It is obvious that $\tilde{P} \in \mathcal{P}(n+1, d)$ (i.e., \tilde{P} is an $(n+1, d)$ -abstract polytope) and that $\delta(\tilde{P}) \geq \delta(P)$, hence

$$\Delta_a(n+1, d) \geq \delta(\tilde{P}) \geq \delta(P) = \Delta_a(n, d).$$

(iii) Let $A'_i \notin UP$ ($i = 1, 2$) be distinct new symbols. Define $P_i = \{(v \cup A'_i) | v \in P\}$, $i = 1, 2$. Then $P_1 \cup P_2 \in \mathcal{P}(n+2, d+1)$ and $\delta(P_1 \cup P_2) = \delta(P) + 1$. So

$$\Delta_a(n+2, d+1) - 1 \geq \delta(P_1 \cup P_2) - 1 = \delta(P) = \Delta_a(n, d).$$

(iv) Let $P \in \mathcal{P}(d+k, k)$ ($k \geq d$) where $\delta(P) = \Delta_a(d+k, k)$. Choose $v, \bar{v} \in P$ so that the shortest path from v to \bar{v} has length $\delta(P)$. Consider the face $P' = F_P(v \cap \bar{v})$ of P which corresponds 1-1 to a $(d+k', k')$ -abstract polytope (where $k' = k - |v \cap \bar{v}| \leq d$). Since $P' \subset P$, the length of the shortest path from v to \bar{v} in P' is at least as large as $\delta(P)$. Hence

$$\Delta_a(d+k', k') \geq \delta(P') \geq \delta(P) = \Delta_a(d+k, k) .$$

However by (i) (since $d+k \geq 2d \geq d+k'$)

$$\Delta_a(d+k, k) \geq \Delta_a(2d, d) \geq \Delta_a(d+k', k')$$

Hence

$$\Delta_a(d+k, k) = \Delta_a(2d, d) \quad \text{for } k \geq d. \quad \square$$

We shall make frequent use of the notion of a "shell" bordering a set of vertices of an abstract polytope. Let P be an abstract polytope and let $Z \subset P$. A vertex v of P belongs to the 1-th shell $N_P^1(Z)$ of Z in P if, and only if 1 is the minimum length of all of the paths in P joining v to the various vertices of Z . The 0-shell of Z is Z itself. The 1-shell of Z is the set of vertices which are adjacent to but not in Z . In general

$$N_P^1(Z) = N_P^1\left(\bigcup_{j=1}^{i-1} N_P^j(Z)\right) .$$

For simplicity, the 1-shell of Z in P will also be denoted by $N_P(Z)$ or simply $N(Z)$ if P is clear.

Theorem 3 below will be used (in section 3) to establish the values of $\Delta_a(n,2)$ and the values of $\Delta_a(n,d)$ for all n, d such that $n-d \leq 5$.

Theorem 3. Given $P \in \mathcal{P}(2d,d)$ (i.e., P is a $(2d,d)$ -abstract polytope) and $v_0, \bar{v}_0 \in P$ such that v_0, \bar{v}_0 partition $U P$. Let (v_0, v_1, \dots, v_k) and $(\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{\bar{k}})$ be two paths in P with the property $|v_1 \cap \bar{v}_j| = i+j$, then such paths exist for

- (i) $d \geq 1, k = 0, \bar{k} = 1$ where \bar{v}_1 is any given vertex in $N_P(\bar{v}_0)$.
- (ii) $d \geq 2, k = 1, \bar{k} = 1$ where \bar{v}_1 is any given vertex in $N_P(\bar{v}_0)$.
- (iii) $d \geq 3, k = 2, \bar{k} = 1$ where \bar{v}_1 is any given vertex in $N_P(\bar{v}_0)$.
- (iv) $d \geq 4, k = 2, \bar{k} = 2$.

Proof. (Except part (iv) for $d \geq 5$.)

Let $v_0 = \{A_1, \dots, A_d\}, \bar{v}_0 = \{\bar{A}_1, \dots, \bar{A}_d\}$ partition P . The symbols v_1, \bar{v}_j where used below satisfy $|v_1 \cap \bar{v}_j| = i+j$ or will be shown to do so.

(i) Obvious by the second axiom of abstract polytopes.

(ii) By (i), $\bar{v}_1 \in N(\bar{v}_0)$ implies that $|v_0 \cap \bar{v}_1| = 1$. Reliable so that $\bar{v}_1 = \{A_1, \bar{A}_1, \dots, \bar{A}_{d-1}\}$. Note that $\bar{A}_d \notin \{v_0 \cup \bar{v}_1\}$. By Theorem 1 there exists an \bar{A}_d -avoiding path between v_0 and \bar{v}_1 in $F(v_0 \cap \bar{v}_1)$ (i.e., all the vertices of that path contain $v_0 \cap \bar{v}_1 = A_1$ but do not contain \bar{A}_d). Let v_1 be the neighbor of v_0 in this path. Since $\bar{A}_d \notin v_1, v_1$ must contain, for $d \geq 2$, one symbol different from those in $v_0 \cup \bar{A}_d$. Hence, since $A_1 \subset v_1, |v_1 \cap \bar{v}_1| = 2$.

(iii) Let $\bar{v}_1 \in N(\bar{v}_0)$ then by (ii) there exists a vertex $v_1 \in N(v_0)$ such that $|v_1 \cap \bar{v}_1| = 2$. By relabeling let $v_1 = (A_1, \dots, A_{d-1}, \bar{A}_1)$, $\bar{v}_1 = (A_1, \bar{A}_1, \dots, \bar{A}_{d-1})$. Define $P' = F(v_1 \cap \bar{v}_1)$ and $W = N(v_0) \cap P'$. Note that W is the set of all vertices of $N(v_0)$ which contain both A_1 and \bar{A}_1 .

By Theorem 1 there exists an \bar{A}_d -avoiding path from v_1 to \bar{v}_1 in P' . Let v_2 be a vertex of this path which belongs to $N_{P'}(W)$ (such vertex exists because $v_1 \in W$ while $\bar{v}_1 \notin W$ for $d \geq 3$). But v_2 contains (A_1, \bar{A}_1) and one symbol out of $\bar{A}_2, \dots, \bar{A}_{d-1}$, hence $|v_2 \cap \bar{v}_1| = 3$.

(iv) ($d = 4$) By (iii) there exists $\bar{v}_1 \in N(\bar{v}_0)$ and $v_2 \in N^2(v_0)$ such that $|v_2 \cap \bar{v}_1| = 3$. Since $d = 4$, the second axiom of abstract polytopes implies that v_2 is a neighbor of \bar{v}_1 . Thus letting $\bar{v}_2 = v_2$ completes the proof for this case. \square

Part (iv) for $d = 5$ will be established in stages via Theorems 4 and 5 and for $d \geq 6$ via Theorems 6 and 7.

Theorem 4. Given $P \in \mathcal{P}(2d, d)$ (where $d \geq 4$) and vertices $v_0, v_1, \bar{v}_0 \in P$ satisfying $|v_1 \cap \bar{v}_1| = i+j$ and $\bar{v}_2', \bar{v}_2'' \in N^2(\bar{v}_0)$ satisfying $|v_1 \cap \bar{v}_2' \cap \bar{v}_2''| = 3$ then there exist $v_2 \in N^2(v_0)$ such that $|v_2 \cap \bar{v}_2| = 4$ where either $\bar{v}_2 = \bar{v}_2'$ or $\bar{v}_2 = \bar{v}_2''$.

Proof: By relabeling for $d \geq 4$ we are assuming

$$v_0 = (A_1, \dots, A_d); \quad \bar{v}_0 = (\bar{A}_1, \dots, \bar{A}_d); \quad v_1 = (A_1, \dots, A_{d-1}, \bar{A}_1);$$

$$\bar{v}_2' = (A_1, A_2, \bar{A}_1, \dots, \bar{A}_d) - (\bar{A}_1, \bar{A}_j);$$

$$\bar{v}_2'' = (A_1, A_2, \bar{A}_1, \dots, \bar{A}_d) - (\bar{A}_k, \bar{A}_\ell) \quad (2 \leq j, k, \ell \leq d)$$

where i, j, k, ℓ are all distinct or $i = k$ and i, j, ℓ are distinct.

Note that $v_1 \cap \bar{v}_2' \cap \bar{v}_2'' = (A_1, A_2, \bar{A}_1)$.

By Theorem 1 there exists an \bar{A}_1 -avoiding path from v_1 to \bar{v}_2' in $P' = F(v_1 \cap \bar{v}_2')$. Let $Z = N(v_0) \cap P'$. Since $v_1 \in Z$, while $\bar{v}_2' \notin Z$, this path intersects $N_{P'}(Z)$, say at v_2 . Note that $v_2 \in N_{P'}(Z) \subset N_P^2(v_0)$.

By definition, all the vertices of $P' = F(v_1 \cap \bar{v}_2')$ contain the symbols A_1, A_2 and \bar{A}_1 and v_2 contains also some $\bar{A}_t \in (\bar{A}_t | t \in [2, \dots, d], t \neq 1)$. But, either \bar{v}_2' or \bar{v}_2'' must contain \bar{A}_t . Hence either $\bar{v}_2 = \bar{v}_2'$ or $\bar{v}_2 = \bar{v}_2''$ satisfies $|v_2 \cap \bar{v}_2| = 4$. \square

Theorem 5. Let $P \in \mathcal{P}(10, 5)$ and let $(v_0, v_1, v_2); (\bar{v}_0, \bar{v}_1)$ be paths in P satisfying $|v_1 \cap \bar{v}_1| = i+j$. Define $P' = F(v_1 \cap \bar{v}_1)$, $W = N(v_0) \cap P'$ and $\bar{W} = N(\bar{v}_0) \cap P'$. Then either there exists a path of length 3 connecting a vertex in W to a vertex in \bar{W} or:

- (a) $F(v_2 \cap \bar{v}_1)$ is (by relabeling) the 7-vertex 2-dimensional abstract polytope given in Figure 1.

$$v_2 = \left[\begin{array}{l} (A_1, A_2, A_3, \bar{A}_1, \bar{A}_2) - (A_1, A_3, A_5, \bar{A}_1, \bar{A}_2) - (A_1, A_5, \bar{A}_1, \bar{A}_2, \bar{A}_4) \\ (A_1, A_2, \bar{A}_1, \bar{A}_2, \bar{A}_5) - (A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_5) - (A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3) \end{array} \right] \searrow \nearrow (A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4) = \bar{v}_1$$

Figure 1

- (b) $F(v_1 \cap \bar{v}_1)$ is (by relabeling) the 3-dimensional abstract polytope given in Figure 2 (note that the graph of $F(v_1 \cap \bar{v}_1)$ is non-planar).

Proof: Assume (by relabeling if necessary) that

$$v_0 = \{A_1, \dots, A_d\}, v_1 = \{A_1, \dots, A_{d-1}, \bar{A}_1\}, v_2 = \{A_1, \dots, A_{d-2}, \bar{A}_1, \bar{A}_2\}, \\ \bar{v}_0 = \{\bar{A}_1, \dots, \bar{A}_d\}, \bar{v}_1 = \{A_1, \bar{A}_1, \dots, \bar{A}_{d-1}\}.$$

- (a) Since $|v_2 \cap \bar{v}_1| = 3$, $P'' = F(v_2 \cap \bar{v}_1)$ corresponds 1-1 to an $(n, 2)$ -abstract polytope with $3 \leq n \leq 7$. It is easy to show that every $(n, 2)$ -abstract polytope Q has exactly n vertices and that every symbol of UQ is contained by two adjacent vertices of Q . Furthermore, the graph of Q forms a simple cycle. We shall consider three cases.
- (a1) $|P''| \leq 5$. Obviously there exists a path of length less than or equal to 3 joining v_1 in W to \bar{v}_1 in \bar{W} .
- (a2) $|P''| = 6$. In this case P'' has the form:

$$v_2 = (A_1, A_2, A_3, \bar{A}_1, \bar{A}_2) \begin{array}{c} \nearrow v'_3 - v'_4 \\ \searrow v''_3 - v''_4 \end{array} (A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4) = \bar{v}_1$$

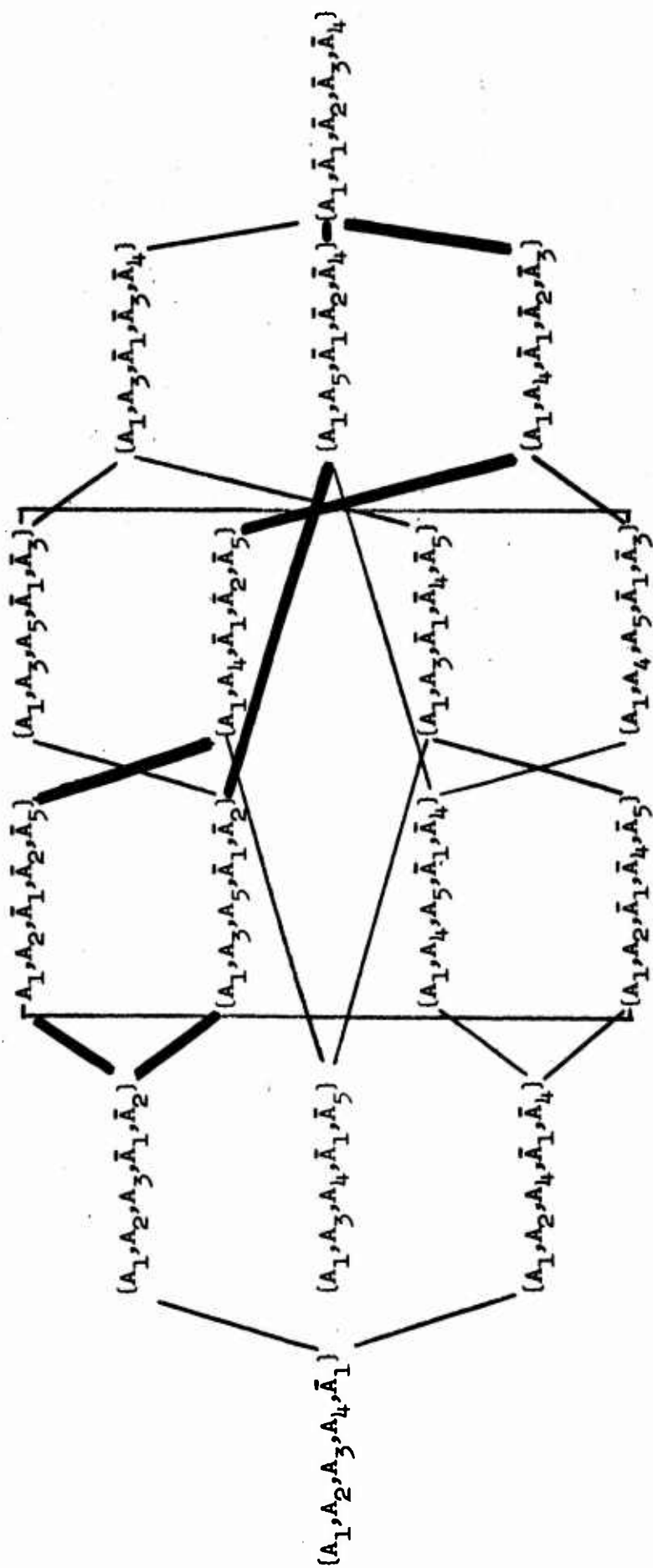
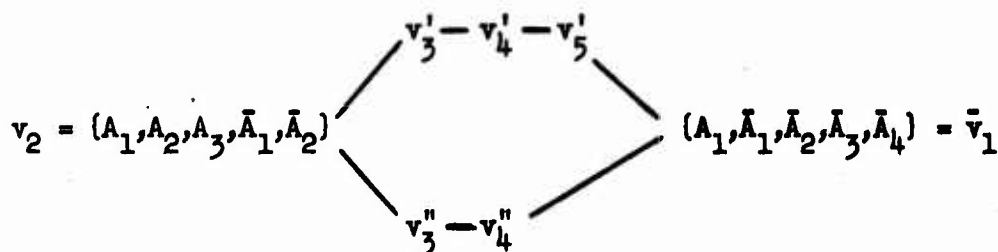


Figure 2

Since A_1, \bar{A}_1 and \bar{A}_2 are contained by all the vertices of P'' , and $v_2 \cup \bar{v}_1 = (A_1, A_2, A_3, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, one of the remaining symbols A_4, A_5 or \bar{A}_5 is contained by the two adjacent vertices v_3^i, v_4^i and another by v_3^i, v_4^i . But if A_4 is contained by v_3^i, v_4^i (or v_3^i, v_4^i) then v_3^i (or v_3^i) is a neighbor of v_1 . If \bar{A}_5 is contained by v_3^i, v_4^i (or v_3^i, v_4^i) then v_4^i (or v_4^i) is a member of \bar{W} . In both cases there exists a path of length 3 from a member of W to a member of \bar{W} .

(a3) $|P''| = 7$. Here P'' has the form:



Using the same arguments as in (a2) we see that if every path from a member of W to a member of \bar{W} has a length of at least 4 then v_3^i, v_4^i must contain \bar{A}_5 , v_4^i, v_5^i must contain A_4 and v_3^i, v_4^i must contain A_5 . Thus P'' has the form of Figure 1 except for possible interchange of symbols A_2 with A_3 and \bar{A}_3 with \bar{A}_4 .

(b) Suppose every path in P' joining a member of W to a member of \bar{W} has a length larger than 3. So P'' has the form given in Figure 1. Let us denote $\bar{v}_2 = (A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3)$ we can apply now the above analysis to the face $F(v_1 \cap \bar{v}_2)$ where we replace the symbols

$\{A_1, A_2, A_3, A_4, A_5\}$ above by $\{\bar{A}_1, \bar{A}_3, \bar{A}_2, \bar{A}_4, \bar{A}_5\}$, and $\{\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5\}$ by $\{A_1, A_4, A_2, A_3, A_5\}$ respectively. Thus $F(v_1 \cap \bar{v}_2)$ has the following form (with the possible interchanging of A_2, A_3 and \bar{A}_2, \bar{A}_3):

$$v_1 = \{A_1, A_2, A_3, A_4, \bar{A}_1\} \left\{ \begin{array}{l} \{A_1, A_3, A_4, \bar{A}_1, \bar{A}_5\} - \{A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_5\} - \{A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3\} \\ \{A_1, A_2, A_4, \bar{A}_1, \bar{A}_4\} - \{A_1, A_4, A_5, \bar{A}_1, \bar{A}_4\} - \{A_1, A_4, A_5, \bar{A}_1, \bar{A}_3\} \end{array} \right\} = \bar{v}_2$$

Note that interchanging \bar{A}_2, \bar{A}_3 is not possible since $\{A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_5\}$ is forced as a neighbor of \bar{v}_2 because it already exists as a vertex of $F(v_2 \cap \bar{v}_1)$

We now let $v'_2 = \{A_1, A_2, A_4, \bar{A}_1, \bar{A}_4\}$ and apply the argument of (a) on $F(v'_2 \cap \bar{v}_1)$. Since $\{A_1, A_5, \bar{A}_1, \bar{A}_2, \bar{A}_4\} \in F(v_2 \cap \bar{v}_1)$ and $\{A_1, A_4, A_5, \bar{A}_1, \bar{A}_4\} \in F(v_1 \cap \bar{v}_2)$ we get a unique form for $F(v'_2 \cap \bar{v}_1)$ as follows:

$$v'_2 = \left\{ \begin{array}{l} \{A_1, A_2, A_4, \bar{A}_1, \bar{A}_4\} - \{A_1, A_4, A_5, \bar{A}_1, \bar{A}_4\} - \{A_1, A_5, \bar{A}_1, \bar{A}_2, \bar{A}_4\} \\ \{A_1, A_2, \bar{A}_1, \bar{A}_4, \bar{A}_5\} - \{A_1, A_3, \bar{A}_1, \bar{A}_4, \bar{A}_5\} - \{A_1, A_3, \bar{A}_1, \bar{A}_3, \bar{A}_4\} \end{array} \right\} \rightarrow \{A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4\} = \bar{v}_1$$

If we would have interchange A_2 with A_3 in $F(v_1 \cap \bar{v}_2)$ then we would have $\{A_1, A_2, \bar{A}_1, \bar{A}_4, \bar{A}_5\} \in F(v'_2 \cap \bar{v}_1)$ and $\{A_1, A_2, A_4, \bar{A}_1, \bar{A}_5\} \in F(v_1 \cap \bar{v}_2) \subset P$ but $\{A_1, A_2, \bar{A}_1, \bar{A}_2, \bar{A}_5\} \in F(v_2 \cap \bar{v}_1) \subset P$. So axiom (11) of abstract polytopes would have been violated since $\{A_1, A_2, \bar{A}_1, \bar{A}_5\}$ is contained by 3 vertices of P .

Hence, we can conclude that A_2 is not interchangeable with A_3 in $F(v_1 \cap \bar{v}_2)$ (i.e., the structure of this face which is given above is the only structure which is compatible with $F(v_2 \cap \bar{v}_1)$ given in (a) and the assumption that no path of length less than 4 joined a member of W to a member of \bar{W}).

Finally, letting $\bar{v}_2' = \{A_1, A_3, \bar{A}_1, \bar{A}_3, \bar{A}_4\}$, considering $F(v_1 \cap \bar{v}_2')$ and applying (a) (notice that $\{A_1, A_3, A_5, \bar{A}_1, \bar{A}_2\}$; $\{A_1, A_2, A_3, \bar{A}_1, \bar{A}_2\}$; $\{A_1, A_3, A_4, \bar{A}_1, \bar{A}_5\}$ and $\{A_1, A_3, \bar{A}_1, \bar{A}_4, \bar{A}_5\}$ belong to $F(v_1 \cap \bar{v}_2')$) we get that the form of $F(v_1 \cap \bar{v}_2')$ is necessarily as follows:

$$v_1 = \{A_1, A_2, A_3, A_4, \bar{A}_1\} \begin{cases} \{A_1, A_3, A_4, \bar{A}_1, \bar{A}_5\} - \{A_1, A_3, \bar{A}_1, \bar{A}_4, \bar{A}_5\} - \{A_1, A_3, \bar{A}_1, \bar{A}_3, \bar{A}_4\} \\ \{A_1, A_2, A_3, \bar{A}_1, \bar{A}_2\} - \{A_1, A_3, A_5, \bar{A}_1, \bar{A}_2\} - \{A_1, A_3, A_5, \bar{A}_1, \bar{A}_3\} \end{cases} = \bar{v}_2'$$

Collecting all the vertices of the four 2-dimensional faces considered above it is not difficult to verify that they form a 3-dimensional abstract polytope which has the structure described in Figure 2. □

Remark. Klee and Walkup [1] name the property that every path from a member of W to a member of \bar{W} has a length of at most 3, as "Property A". They show that every 3-dimensional simple polytope with 8 facets satisfies property A. We have shown that all 3-dimensional abstract polytopes with 8 facets also satisfy property A except one, namely the one with structure given in Figure 2.

Proof of Theorem 3, Part (iv) for $d = 5$: It is obvious that (iv) holds if and only if there exists a path of length 5 from v_0 to \bar{v}_0 , i.e., if, and only if, there exists a path of length 3 from a neighbor of v_0 to a neighbor of \bar{v}_0 .

Suppose (iv) does not hold, then by Theorem 5-b every 3-dimensional face of P which is generated by a member of $N(v_0)$ and a member of $N(\bar{v}_0)$ has the structure of Figure 2 after relabeling. In particular let $v_1 = \{A_1, A_2, A_3, A_4, \bar{A}_1\}$, $v_2 = \{A_1, A_2, A_3, \bar{A}_1, \bar{A}_2\}$, $\bar{v}_1 = \{A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4\}$ and $P' = F(v_1 \cap \bar{v}_1)$ has the form of Figure 2. Consider $v_0 = \{A_1, A_2, A_3, A_4, A_5\}$ and its incident edge generated by $\{A_1, A_3, A_4, A_5\}$. The other vertex incident to this edge cannot be $\{A_1, A_3, A_4, A_5, \bar{A}_i\}$ where $i = 1, 2, 3, 4$ since this would imply that there is a path of length 5 from v_0 to \bar{v}_0 via that edge and one of the following four vertices of P' :

$\{A_1, A_3, A_5, \bar{A}_1, \bar{A}_3\}$; $\{A_1, A_4, A_5, \bar{A}_1, \bar{A}_3\}$; $\{A_1, A_3, A_5, \bar{A}_1, \bar{A}_2\}$ and $\{A_1, A_4, A_5, \bar{A}_1, \bar{A}_4\}$

Hence $\{A_1, A_3, A_4, A_5, \bar{A}_5\}$ is a vertex adjacent to v_0 .

Similar arguments lead to the conclusion that either

$\{A_1, A_2, A_4, A_5, \bar{A}_5\}$ or $\{A_1, A_2, A_4, A_5, \bar{A}_2\}$ is the other vertex (beside v_0) which is incident to the edge generated by $\{A_1, A_2, A_4, A_5\}$.

The same argument with respect to \bar{v}_0 , the edge generated by $\{\bar{A}_1, \bar{A}_2, \bar{A}_4, \bar{A}_5\}$ and vertices $\{A_1, A_4, \bar{A}_1, \bar{A}_2, \bar{A}_5\}$; $\{A_1, A_3, \bar{A}_1, \bar{A}_4, \bar{A}_5\}$; $\{A_1, A_2, \bar{A}_1, \bar{A}_2, \bar{A}_5\}$; $\{A_1, A_2, \bar{A}_1, \bar{A}_4, \bar{A}_5\}$ in P' implies that $\{A_5, \bar{A}_1, \bar{A}_2, \bar{A}_4, \bar{A}_5\} \in N_P(\bar{v}_0)$. Consider now vertices $\{A_1, A_3, A_4, A_5, \bar{A}_5\}$ and $\{A_5, \bar{A}_1, \bar{A}_2, \bar{A}_4, \bar{A}_5\}$, by Theorem 5 in order not to have a path of length 3 joining these two vertices the face of their

intersection must have the structure of Figure 2. But note in Figure 2 that the three neighbors of v_1 have the property that no two are neighbors. This rules out the possibility that $\{A_1, A_2, A_4, A_5, \bar{A}_5\}$ is a vertex as it would lie in the face and would be a neighbor of $\{A_1, A_3, A_4, A_5, \bar{A}_5\}$. Hence if (iv) does not hold, $\{A_1, A_2, A_4, A_5, \bar{A}_2\} \in N(v_0)$.

Let us now consider the face $F(\{A_1, A_2, A_4, A_5, \bar{A}_2\} \cap \{A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4\}) = F(\{A_1, \bar{A}_2\})$. It contains the non-empty 3-dimensional face $F(\{A_1, A_2, A_3, \bar{A}_1, \bar{A}_2\} \cap \{A_1, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4\}) = F(\{A_1, \bar{A}_1, \bar{A}_2\})$. By Theorem 5, under the assumption that (iv) does not hold, $F(\{A_1, \bar{A}_2\})$ must have the structure of Figure 2. Note that $F(\{A_1, \bar{A}_1, \bar{A}_2\})$ also lies in $F(v_1 \cap \bar{v}_1)$ and has 7 vertices (shown connected by heavy arcs in Figure 2). But all two dimensional faces with seven vertices of the abstract polytope given in Figure 2 have the property that one of its vertices is adjacent to v_1 in P' and thus analogously to $\{A_1, A_2, A_4, A_5, \bar{A}_2\}$ in $F(\{A_1, \bar{A}_2\})$, but in fact none are, a contradiction. So (iv) must hold. \square

The last part of Theorem 3-(iv), for $d \geq 6$, will be proved via Theorems 6 and 7.

Theorem 6. Let $P \in \mathcal{P}(2d, d)$ and let $\{v_0, v_1, v_2\}, \{\bar{v}_0, \bar{v}_1\}$ be two paths in P such that $|v_1 \cap \bar{v}_1| = i+j$. Let $W = N(v_0) \cap F(v_1 \cap \bar{v}_1)$; then if $d \geq 6$ and $W \geq 2$ there exists $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$.

Proof. By relabeling let $v_0 = (A_1, \dots, A_d)$; $v_1 = (A_1, \dots, A_{d-1}, \bar{A}_1)$; $v_2 = (A_1, \dots, A_{d-2}, \bar{A}_1, \bar{A}_2)$; $\bar{v}_0 = (\bar{A}_1, \dots, \bar{A}_d)$, and $\bar{v}_1 = (A_1, \bar{A}_1, \dots, \bar{A}_{d-1})$. Define $P' = F(v_1 \cap \bar{v}_1)$. (Thus $W = N(v_0) \cap P'$); $P'' = F(v_2 \cap \bar{v}_1)$; $\bar{Z} = N(\bar{v}_0) \cap P''$ and $\bar{U}_1 = \{v \in N_{P''}(\bar{Z}) \mid A_1 \subset v\}$. ($i = 2, \dots, d$). The proof of Theorem 6 obviously is a result of the following Lemma [part (b4)].

Lemma.

- (a) $\bar{U}_2, \dots, \bar{U}_d$ partitions $N_{P''}(\bar{Z})$.
- (b) Either there exist $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$ or:
 - (b1) $|\bar{U}_1| = 0$ for $i = 2, \dots, d-2$
 - (b2) $|\bar{U}_{d-1}| = 1$
 - (b3) $|\bar{U}_d| \geq d-4$
 - (b4) $|W| = 1$ for $d \geq 6$.

Proof of the Lemma.

(a) Since every vertex of P'' contains A_1 and since every vertex of $N_{P''}(\bar{Z})$ contains exactly two non-barred symbols, obviously:

$$N_{P''}(\bar{Z}) = \bigcup_{i=2}^d \bar{U}_i \quad \text{and} \quad \bar{U}_i \cap \bar{U}_j = \emptyset \quad \text{for } i, j = 2, \dots, d, i \neq j.$$

(b1) Assume $\bar{U}_{i_0} \neq \emptyset$ and $\bar{v}'_2 \in \bar{U}_{i_0}$ for some $i_0, 2 \leq i_0 \leq d-2$, then $(A_1, A_{i_0}, \bar{A}_1, \bar{A}_2) \subset \bar{v}'_2$. Hence $|v_2 \cap \bar{v}'_2| = 4$. Moreover, $\bar{v}'_2 \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$.

(b2) By Theorem 1 there exists an A_d -avoiding path joining v_2 to \bar{v}_1 in P^n . Since $\bar{v}_1 \in \bar{Z}$ and $v_2 \notin \bar{Z}$ this path must intersect $N_{P^n}(\bar{Z})$. Thus, there exists a vertex $\bar{v}'_2 \in N_{P^n}(\bar{Z}) \subset N^2(\bar{v}_0)$ which does not contain A_d implying at least one $|\bar{U}_1| \neq 0$ for $i = 2, \dots, d-1$. So by (b1) either there exists $\bar{v}'_2 \in N^2_P(\bar{v}_0)$ such that $|v_2 \cap \bar{v}'_2| = 4$ or $|\bar{U}_{d-1}| > 0$.

Assume now that $|\bar{U}_{d-1}| \geq 2$ and let $\bar{v}'_2, \bar{v}''_2 \in \bar{U}_{d-1}$. Since \bar{v}'_2 and \bar{v}''_2 both contain A_1, A_{d-1} and \bar{A}_1 we have $|v_1 \cap \bar{v}'_2 \cap \bar{v}''_2| = 3$. Furthermore $\bar{v}'_2, \bar{v}''_2 \in N_{P^n}(\bar{Z}) \subset N^2(\bar{v}_0)$, so by Theorem 4 there exists $v'_2 \in N^2(v_0)$ such that either $|v'_2 \cap \bar{v}'_2| = 4$ or $|v'_2 \cap \bar{v}''_2| = 4$.

Thus we conclude that either $|\bar{U}_{d-1}| = 1$ or there exists $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$.

(b3) Suppose $|\bar{Z}| = k$. Note that $k \geq 1$ because $\bar{v}_1 \in \bar{Z}$. The vertices of \bar{Z} have the form $\{A_1, \bar{A}_1, \dots, \bar{A}_d\} - \{\bar{A}_1\}$, $i \in R$, where R is a subset of k indices of $\{3, \dots, d\}$.

By the second axiom of abstract polytopes the subset $\{A_1, \bar{A}_1, \dots, \bar{A}_d\} - \{\bar{A}_1, \bar{A}_j\}$, $(i \in R, j \in \{3, \dots, d\}, j \notin R)$ is contained by two vertices of P^n . Thus every vertex of \bar{Z} gives rise to $d-2-k$ distinct vertices in $N_{P^n}(\bar{Z})$. Therefore $|N_{P^n}(\bar{Z})| = k(d-2-k)$. Hence by (a), (b1) and (b2), either there exists $v'_2 \in N^2(v_0)$, $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$ or

$$|N_{P^n}(\bar{Z})| = |\bar{U}_d| + 1 = k(d-2-k).$$

The last expression implies that

$$0 < k < d-2 \quad \text{and} \quad |\bar{U}_d| \geq d-4.$$

(b4) Finally let us assume that $d \geq 6$ and $|W| \geq 2$ and let $v'_1 \in W$ be distinct from v_1 . Thus $\{A_1, A_d, \bar{A}_1\} \subset v'_1$. By (b3) either there exists $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$ or $|\bar{U}_d| \geq d-4 \geq 2$ for $d \geq 6$. Accordingly let $\bar{v}'_2, \bar{v}''_2 \in \bar{U}_d$. Since both \bar{v}'_2 and \bar{v}''_2 contain $\{A_1, A_d, \bar{A}_1\}$ we have $|v'_1 \cap \bar{v}'_2 \cap \bar{v}''_2| = 3$ which by Theorem 4 implies either $|v_2 \cap v'_2| = 4$ or $|v_2 \cap \bar{v}''_2| = 4$, which completes the proof of the Lemma. \square

Theorem 7: $d \geq 6, k = 2, \bar{k} = 2$ holds for Theorem 3, if there exist

$$v'_1, v''_1 \subset N(v_0) \quad \text{and} \quad \bar{v}_1 \subset N(\bar{v}_0) \quad \text{such that} \quad |v'_1 \cap v''_1 \cap \bar{v}_1| = 2.$$

Proof. Without loss of generality we can assume that

$$v_0 = \{A_1, \dots, A_d\}; \quad v_1 = \{A_1, \dots, A_{d-1}, \bar{A}_1\}; \quad v'_1 = \{A_1, \dots, A_{d-2}, A_d, \bar{A}_1\};$$

$$\bar{v}_1 = \{A_1, \bar{A}_1, \dots, \bar{A}_{d-1}\}; \quad \bar{v}_0 = \{\bar{A}_1, \dots, \bar{A}_d\}.$$

We wish to show that if $d \geq 6$ then there exist $v'_2 \in N^2_P(v_0)$ and $\bar{v}'_2 \in N^2_P(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$.

By Theorem 1 there exists an \bar{A}_d -avoiding path from v_1 to \bar{v}_1 in $P' = F(v_1 \cap \bar{v}_1)$. This path intersects $N_P(W)$ at, say, v_2 . In this case $v_0, v_1, v_2, \bar{v}_0, \bar{v}_1$ satisfy the conditions of Theorem 6. Moreover, $|W| \geq 2$ since $v_1, v'_1 \subset W$ and $d \geq 6$ so that by Theorem 6 there exist $v'_2 \in N^2_P(v_0)$ and $\bar{v}'_2 \in N^2_P(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$. \square

Proof of Theorem 3 Part (iv) for $d \geq 6$: By Theorem 3 (iii)

we can assume the existence of paths $(v_0, v_1, v_2); (\bar{v}_0, \bar{v}_1)$ such that

$|v_1 \cap \bar{v}_1| = 1+j$. Without loss of generality we can assume that

$$v_0 = \{A_1, \dots, A_d\}; \quad v_1 = \{A_1, \dots, A_{d-1}, \bar{A}_1\}; \quad v_2 = \{A_1, \dots, A_{d-2}, \bar{A}_1, \bar{A}_2\} \text{ and} \\ \bar{v}_1 = \{A_1, \bar{A}_1, \dots, \bar{A}_{d-1}\}; \quad \bar{v}_0 = \{\bar{A}_1, \dots, \bar{A}_d\}.$$

Let us define P', P'', W, \bar{Z} and \bar{U}_i ($i = 2, \dots, d$) as in the preceding Lemma. Since we assume that $d \geq 6$ we have, by the Lemma that $|\bar{U}_d| \geq 2$. Let $\bar{v}_2, \bar{v}_2' \in \bar{U}_d$.

If $|\bar{Z}| \geq 2$ then (considering the two vertices in \bar{Z} and v_1) (iv) holds by Theorem 7. If $|\bar{Z}| = 1$, then, necessarily \bar{v}_2, \bar{v}_2' have the form:

$$\bar{v}_2 = \{A_1, A_d, \bar{A}_1, \dots, \bar{A}_{d-1}\} - \{\bar{A}_1\}; \quad \bar{v}_2' = \{A_1, A_d, \bar{A}_1, \dots, \bar{A}_{d-1}\} - \{\bar{A}_j\}$$

for some $i, j : 3 \leq i, j \leq d-1$ and $i \neq j$.

Thus by Theorem 4 either (iv) holds or every vertex of $W' = N_P(v_0) \cap F(v_0 \cap \bar{v}_1)$ (except v_1) contains one symbol out of \bar{A}_1, \bar{A}_j and \bar{A}_d . But since $d \geq 6$, $|W'| = d-1 \geq 5$. Thus, at least two vertices of W' , say v_1', v_1'' are adjacent. But then $|v_1' \cap v_1'' \cap \bar{v}_1| = 2$ and (iv) holds by Theorem 7.

□

Corollary 1.

- (i) $\Delta_a(2d+k, d) \leq \Delta_a(2d+k-1, d-1) + 1, k = 0, 1$
 (ii) $\Delta_a(2d, d) \leq \Delta_a(2d-k, d-k) + k, k = 1, 2, 3, 4.$

Proof:

(i) follows from Theorem 3-(1), (ii).

(ii) follows from Theorem 3. □

Note that since every simple polytope satisfies the axioms of an abstract polytope, Theorem 3 holds for simple polytopes and Corollary 1 holds if one replaces $\Delta_a(n, d)$ by $\Delta_b(n, d)$ (the maximum diameter of ordinary polytopes over all d -dimensional polytopes with n facets).

5. Maximum diameters of abstract polytopes and the Hirsch conjecture.

Hirsch conjecture. Corresponding to the Hirsch conjecture of polytopes, Dantzig [4], is the conjecture for abstract polytopes that

$$\Delta_a(n, d) \leq n - d \quad (d \geq 1, n \geq d+1).$$

Theorem 8 below is the analog of the results of Klee and Walkup [1] for abstract polytopes (except for $\Delta_a(n, 3), n \geq 9$) and is mainly based on Theorem 3.

Theorem 8. The values of $\Delta_a(n,d)$ for $d \leq 2$ and for $n-d \leq 5$ are given in the following table:

$d \backslash n-d$	1	2	3	4	5
2	"	2	2	3	3
3	"	"	3	3	4
4	"	"	"	4	5
$d \geq 5$	"	"	"	"	5

$$\dots \Delta_a(n,2) = \lfloor n/2 \rfloor$$

Table 1: Values of $\Delta_a(n,d)$

(The double quote mark indicates that each column is constant from the main diagonal downwards.)

Proof: As was pointed out in Section 3, $\Delta_a(n,d) \geq \Delta_b(n,d)$. Thus since Table 1 holds for $\Delta_b(n,d)$ (Klee and Walkup [1]), it is sufficient to show that the values in Table 1 are upper bounds for $\Delta_a(n,d)$.

Let $P \in \mathcal{P}(n,d)$.

- (a) $2d > n$: By Theorem 2-(iv) each column of Table 1 is constant from the main diagonal downwards.
- (b) $d = 2, n \geq 4$: Since P is a 2-dimensional abstract polytope, the number of vertices of P is equal to the number of its edges, therefore the graph of P forms a simple cycle with n vertices. Hence $\Delta_a(n,2) = \lfloor n/2 \rfloor$.

(c) $n-d = 3$: By (b) and Corollary 1, $\Delta_a(6,3) = 3$.

(d) $n-d = 4$:

(d1) $n = 7$: If $v \cap \bar{v} \neq \emptyset$ for every pair of vertices $v, \bar{v} \in P$ then $F(v \cap \bar{v})$ is an (n', d') -abstract polytope with $n' \leq 6$ and $d' \leq 2$.

Thus, by (b) and (c), $\delta(P) \leq 3$.

Suppose now, that there exist $v, \bar{v} \in P$ such that $v \cap \bar{v} = \emptyset$.

Let $UP - \{v \cup \bar{v}\} = A$ then by Theorem 1 there exists an A avoiding path between v and \bar{v} . This path intersects $N^2(v)$, say at v_2 . Since every vertex in $N^2(v)$ contains two symbols of $UP - v$, v_2 is necessarily adjacent to \bar{v} . Hence $\Delta(7,3) \leq 3$.

(d2) $n = 8$: By (d1) and Corollary 1, $\Delta_a(8,4) \leq 4$.

(e) $n-d = 5$:

(e1) $n = 8$: If $v \cap \bar{v} \neq \emptyset$ for all $v, \bar{v} \in P$ then $P' = F(v \cap \bar{v})$ is d' -dimensional face of P where $d' \leq 2$ and $|UP - \{v \cap \bar{v}\}| \leq 7$, thus by (a) and (b), $\delta(P) \leq 3$.

Suppose there exists $v, \bar{v} \in P$ such that $v \cap \bar{v} = \emptyset$. Let

$UP = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$, $v = \{A_1, A_2, A_3\}$ and $\bar{v} = \{A_4, A_5, A_6\}$.

If $\delta(P) > 4$ then every vertex in $N(v)$ and $N(\bar{v})$ contains either A_7 or A_8 (otherwise a vertex in $N(v)$ (or in $N(\bar{v})$) and \bar{v} (or v), both contain the same symbol which, by (iii) implies that $\delta(P) \leq 4$).

Without loss of generality we can assume that

$$N(v) = \{(A_1, A_2, A_7); (A_1, A_3, A_7); (A_2, A_3, A_8)\}$$

and either

$$N(\bar{v}) = \{(A_4, A_5, A_7); (A_4, A_6, A_7); (A_5, A_6, A_8)\}$$

or

$$N(\bar{v}) = \{(A_4, A_5, A_7); (A_4, A_6, A_8); (A_5, A_6, A_8)\}.$$

In the first case the graph of $F(\{A_7\})$ forms a simple cycle with at most 7 vertices while in the second case it forms a simple cycle with at most 6 vertices. (Since in the second case $F(\{A_6\}) =$

$\{(A_4, A_5, A_6), (A_4, A_6, A_8), (A_5, A_6, A_8)\}$ which implies that $F(\{A_7\}) \cap F(\{A_6\}) = \emptyset$.)

In both cases $\delta(P) \leq 4$. Hence, $\Delta_8(8, 3) \leq 4$.

(e2) $n = 9$: By (e1) and Corollary 1, $\Delta_8(9, 4) \leq 5$.

(e3) $n = 10$: If $v \cap \bar{v} \neq \emptyset$ for all $v, \bar{v} \in P$ then $F(v \cap \bar{v})$ is an

(n', d') -abstract polytope with $n' \leq 9$ and $d' \leq 4$. Therefore,

by (e2), $\delta(P) \leq 5$. Suppose, now, that there exist v_0, \bar{v}_0 in P

such that $v_0 \cap \bar{v}_0 = \emptyset$. Without loss of generality we can assume

that $v_0 = (A_1, A_2, A_3, A_4, A_5)$, $\bar{v}_0 = (A_1, A_2, A_3, A_4, A_5)$. Then by Theorem 3-(1v)

$\delta(P) = 5$.

□

Bibliography

- [1] V. Klee and D. W. Walkup, The d -step conjecture for polyhedra of dimension $d < 6$, Acta Math., 117 (1967), 53-78.
- [2] B. Grünbaum, Convex Polytopes, John Wiley and Sons, New York, 1967.
- [3] I. Adler, G. B. Dantzig and K. Murty, Existence of x -path in abstract polytopes, Technical Report No. 70-1, Department of Operations Research, Stanford University (1970).
- [4] G. B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, New Jersey, 1963.